

# An isoperimetric inequality for planar triangulations

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## Abstract

We prove a discrete analogue to a classical isoperimetric theorem of Weil for surfaces with non-positive curvature. It is shown that hexagons in the triangular lattice have maximal volume among all sets of a given boundary in any triangulation with minimal degree 6.

## 1 Introduction

In 1926 Weil [7, 8] proved the following:

**Theorem 1** (Weil). *If  $M$  is a 2-dimensional manifold homeomorphic to the unit disc with non-positive curvature at every point, then*

$$|\partial M| \geq 2\sqrt{\pi|M|}.$$

Thus a disc in the Euclidean plane solves the isoperimetric problem, not just among sets in the plane but also among all surfaces with non-positive curvature. We give a discretized version of this:

**Theorem 2.** *Any disc triangulation with  $V$  vertices and  $n$  boundary vertices, and with all internal degrees at least 6 has  $V \leq \left\lfloor \frac{(n+3)^2}{12} \right\rfloor$ . Equality is achieved in the Euclidean triangular lattice.*

We remark that unlike the proofs in [7, 8], our arguments are purely combinatorial and do not use conformal geometry. We discuss below several consequences and variants of the argument, including for hyperbolic triangulations.

## 2 Proof

*Proof of Theorem 1.* If  $V \leq 6$  then there can be no internal vertices, and the claim clearly holds. We assume henceforth that  $V \geq 6$ . Let  $E, F$  be the number of edges and faces in the triangulation. By Euler's formula,  $V - E + F = 1$  (since the external face is not counted). Let  $n = |\partial T|$ . Counting faces with a marked edge gives  $3F = 2E - n$ , and thus  $E = 3V - n - 3$ .

Let  $\sigma \geq 6$  be the average degree of internal vertices, and let  $M$  be the set of internal edges, incident to the boundary, with a marked endpoint on the boundary, and  $m = |M|$  its cardinality (so that an internal edge between two boundary vertices is counted twice). Summing vertex degrees gives

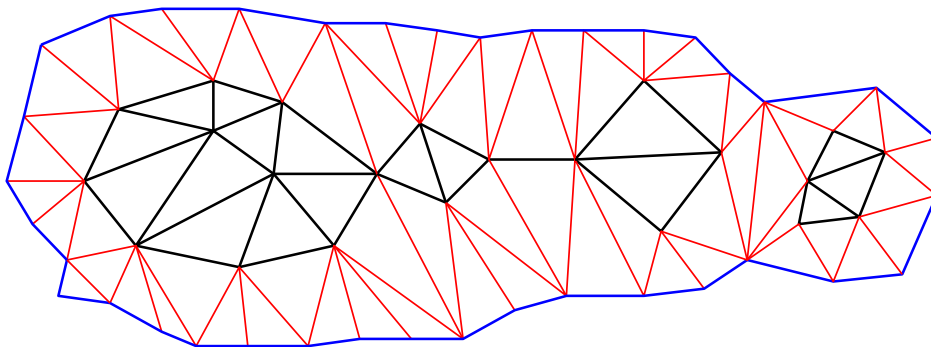
$$2E = \sigma(V - n) + 2n + m,$$

which gives

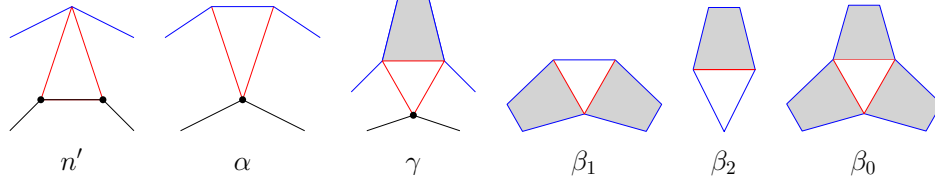
$$m = 2n - 6 - (\sigma - 6)(V - n),$$

and since  $\sigma \geq 6$  and  $V - n$  is the number of internal vertices,  $m \leq 2n - 6$ .

Stripping all boundary vertices and all faces incident to the boundary leaves a smaller triangulation, which need not be a disc triangulation: It may have several connected components, and may have components with a non-simple boundary, containing a cut vertex, or even bridges (See fig. 1). Let  $n'$  be the number of boundary edges of the stripped triangulation (with bridges counted twice). Our objective is to show that  $n' \leq n - 6$ .



**Figure 1:** A disc triangulation with a boundary of length 37. Stripping faces adjacent to the boundary leaves a smaller triangulation, with total boundary length  $23 \leq 37 - 6$ .



**Figure 2:** Different types of faces being stripped, and the number of such faces. Internal vertices are marked. The shade areas may contain any number of faces and vertices.

We now count edges of  $M$  with a marked side (a face incident to them). The total is  $2m$ , and we count it by considering the different types of stripped faces. Each edge of the new boundary (of which there are  $n'$ ) must have a third vertex on the outer boundary, and contains two edges of  $M$ . All other stripped faces have two or three vertices on the outer boundary. Let  $\alpha$  be the number of faces with a boundary edge and internal vertex. Let  $\gamma$  be the number of faces with two boundary vertices and one internal vertex, but no boundary edges. Let  $\beta_i$  be the number of faces with three boundary vertices and  $i$  boundary edges (see fig. 2). Then we have the following identity:

$$2m = 2n' + 2\alpha + 4\gamma + 6\beta_0 + 4\beta_1 + 2\beta_2,$$

as well as

$$n = \alpha + \beta_1 + 2\beta_2 + 3\beta_3.$$

Combining these identities with  $m \leq 2n - 6$  gives

$$n' + 2\gamma + 3\beta_0 + \beta_1 - \beta_2 - 3\beta_3 \leq n - 6.$$

Observe that if  $\beta_3$  is non-zero, the entire triangulation consists of a single face, a case we already dealt with. Otherwise, we need to show  $\beta_2 \leq \beta_1 + 3\beta_0 + 2\gamma$ .

A face contributing to  $\beta_2$  has a unique internal edge with both endpoints on the boundary. The face  $f$  on the other side of this edge will contribute to  $\beta_1, \beta_3$ , or  $\gamma$ .

If  $f$  has an internal third vertex, it counts towards  $\gamma$ . Alternatively,  $f$  may have its third vertex on the boundary. If  $f$  is of type  $\beta_2$ , then the entire triangulation has 4 vertices, all on the boundary. If this face is of type  $\beta_0$  or  $\beta_1$  then it also contributes to  $\beta_1 + 3\beta_0 + 2\gamma$ . A face of type  $\beta_0$  can correspond to at most 3 faces of type  $\beta_2$ . It is possible for a face of type  $\beta_1$  to correspond

to two faces of type  $\beta_2$ , but this only happens if the entire triangulation has 5 vertices, a case we already considered. Thus  $n' \leq n - 6$  as claimed.

Separating the stripped triangulation into a collection of disc triangulations with total boundary size  $n'$ , and proceeding by induction, we find that at distance  $k$  from the boundary there are at most  $n - 6k$  vertices. and summing this gives the volume of a Euclidean hexagon as a bound on  $|T|$ . It follows that the overall number of vertices is bounded by

$$V \leq \sum_{k \geq 0} (n - 6k)^+ = \left\lfloor \frac{(n + 3)^2}{12} \right\rfloor.$$

(The last identity is easily verified by checking cases for  $n \pmod 6$ ).

In the case of a convex hexagon in the triangular lattice we indeed have  $n' = n - 6$ , unless the hexagon has width 1, in which case  $m = 0$ . It follows that equality is achieved by convex hexagons in the triangular lattice with all boundary segments of suitable lengths in  $\{k, k + 1, k + 2\}$ . For example, if  $n = 6k + 7$  the boundary segments have lengths  $(k, k + 2, k + 1, k + 1, k + 1, k + 2)$  in order.  $\square$

Since any map with no multiple edges can be made into a triangulation by adding edges, it follows that the same inequality holds for any simple map in a disc.

**Corollary 3.** *Any simple map in a disc with  $V$  vertices and  $n$  boundary vertices, and with all internal degrees at least 6 has  $V \leq \left\lfloor \frac{(n+3)^2}{12} \right\rfloor$ .*

The edge boundary is also minimized by hexagons in the triangular lattice.

**Theorem 4.** *If  $T$  is a triangulation where all vertices have degree at least 6, and  $A$  is any finite set in  $T$  of size  $V$ , then the edge boundary  $\partial A$  satisfies  $|\partial A| \geq \sqrt{48V}$ .*

*Proof.* Let  $n$  be the number of boundary vertices in  $A$ . From Theorem 2 we have that  $12V \leq (n + 3)^2$ . The number of directed edges from these  $n$  vertices into the triangulation is at most  $2n - 6$  (as in the proof above). Since there are at most  $2n$  directed edges in the boundary (less if it contains any bridges) we have that the number of outward edges is at least  $2n + 6 \geq \sqrt{48V}$ .  $\square$

### 3 Hyperbolic lattices

In [4] it is proved that in the hyperbolic lattice balls solve the Isoperimetric problem. This involves using  $m = 2n - 6 - (\sigma - 6)(V - n)$  with a better bound on the last term. Let  $V_R$  denote the number of vertices in a ball of radius  $R$  in the  $\delta$ -regular triangulation, and let  $S_R$  denote the size of its boundary. (These grow exponentially; Explicit formulae are available but not helpful for us.)

We have the following result, in a sense stating that hyperbolic balls are optimal among all triangulations with minimal degree  $\delta$ .

**Theorem 5.** *If  $T$  is a disc triangulation with minimal degree  $\delta$  and  $V \geq V_R$  vertices. Then  $n \geq S_R$ .*

*Proof.* Repeating the argument above when all vertex degrees are at least some  $\delta > 6$ , we find that after stripping a layer, the new boundary  $n'$  satisfies

$$n' \leq n - 6 - (\delta - 6)(V - n).$$

In the case of a ball in the  $\delta$ -uniform triangulation, there are no faces of type  $\beta_i$ , and there is equality here.

Suppose for a contradiction that there exists a triangulation with  $V \geq V_R$  and  $n < S_R$ , and consider the example with minimal possible  $Rf$ . Then  $V - n > V_R - S_R = V_{R-1}$ , and  $n' < S_R - 6 - (\delta - 6)(V_R - S_R) = S_{R-1}$ . Thus stripping a layer gives a smaller counter-example.  $\square$

**Problem 1.** *Is it true that for any triangulation with minimal degree  $\delta$ , there exists a subset of the  $\delta$ -regular triangulation with the same volume and equal or smaller boundary?*

Finally, we remark that hyperbolicity follows once there are enough vertices of degree greater than 6.

**Proposition 6.** *For every  $R > 0$  there exists  $\alpha > 0$  such that the following holds. Let  $T$  be an infinite plane triangulation with minimal degree 6, and such that for some  $R$ , every ball of radius  $R$  contains a vertex of degree at least 7. Then  $T$  is non-amenable, with isoperimetric constant at least  $\alpha$ .*

*Proof.* First, we may assume that  $T$  has bounded degrees. Indeed, we may replace each vertex of degree greater than 9 by several vertices of degree at

least 7, while keeping all other degrees at least 6. This maintains or reduces the distance to a high degree vertex from any point.

Consider a finite set  $S \subset T$  of size  $V$ . To optimize the isoperimetric ratio, we may assume  $S$  is connected. Let  $\sigma$  be the average degree of internal vertices in  $S$ . Then we have as above  $m = 2n - 6 - (\sigma - 6)(V - n)$ .

For some  $\varepsilon > 0$  to be fixed later, we argue as follows. If  $\sigma \geq 6 + \varepsilon$  this implies the boundary is proportional to  $V$ . If  $\sigma < 6 - \varepsilon$ , then there are at most  $\varepsilon V$  vertices of degree greater than 6. Since degrees are bounded by 9, each of these is within distance  $R$  of at most  $9^R \varepsilon$  vertices, and so  $(1 - 9^R \varepsilon)V$  vertices are within distance  $R$  of  $|\partial S|$ . This implies  $|\partial S| \geq cV$  for some  $c$ , provided  $9^R \varepsilon < 1$ .  $\square$

It should be possible to relax the condition of minimal degree 6 to the following. Suppose there is a partition of the vertices into sets  $A_i$  such that each  $A_i$  has diameter at most  $R$  and such that within each  $A_i$  the average degree is greater than 6.

We remark that the proof above gives  $\alpha > e^{-c/\varepsilon}$ . The proof can be adapted to have  $\alpha$  polynomial in  $R$ .

**Problem 2.** *For any given  $R$ , what is the optimal isoperimetric constant for a triangulation as above?*

## 4 Further questions

Kleiner (1992) [5] proved an analogue of Weil's Theorem for three dimensional manifolds of non-positive curvature, and Croke (1984) [2] proved a four dimensional analogue. The question in all higher dimensions is still open.

It would be interesting to have a three (or higher) dimensional version of our theorems. A specific problem in three (or more) dimensions is as follows.

**Problem 3.** *Show that for every CAT(0) cubulation of a ball with  $V$  cubes and  $A$  boundary squares there is a subset of cubes in the cubic lattice of  $\mathbb{R}^3$  of size  $V$  and at most  $A$  boundary squares?*

Here, a cubulation of a ball is a collection of topological cubes, glued along faces to get a topological ball. There are various possible analogues to the having minimal degree 6. Being CAT(0) is a well studied property of general metric spaces, analogous to having non-positive curvature. The

Cartan-Hadamard Theorem (see e.g. [1, Theorem II.4.1]) implies that a plane triangulation is CAT(0) if and only if it has minimal degree at least 6.

It is similarly possible to characterize a CAT(0) cubulation in terms of the possible local structures at each vertex, and in particular it is necessary (though not sufficient) that each vertex to be in at least 8 cubes. We refer the reader to [1] for the theory of CAT(0) metric spaces, and in particular Theorems II.5.2 and II.5.18 for the characterization in terms of local structure.

Finally, it is also possible for the regular triangulations to be extremal in other ways.

**Problem 4.** *If  $T$  is a plane triangulation with all vertex degrees at least 6, is it necessarily true that the connective constant satisfies  $\mu(T) \geq \mu(\mathbb{T}_6)$ ? Is it true that  $p_c^{\text{site}}(T) \leq 1/2$ , and  $p_c^{\text{bond}}(T) \leq 2 \sin(\pi/18)$ ?*

$1/2$  and  $2 \sin(\pi/18)$  are the values for the 6-regular triangular lattice  $\mathbb{T}_6$  [6]. Similarly, for other  $k$ , do the  $k$ -regular triangulation have the minimal connective constant and maximal percolation threshold.

## References

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